0039-3681(94)00034-4

Strategies for Conceptual Change: Ratio and Proportion in Classical Greek Mathematics

Paul Rusnock and Paul Thagard**

...all men begin... by wondering that things are as they are...as they do about...the incommensurability of the diagonal of the square with the side; for it seems wonderful to all who have not yet seen the reason, that there is a thing which cannot be measured even by the smallest unit. But we must end in the contrary and, according to the proverb, the better state, as is the case in these instances too when men learn the cause; for there is nothing which would surprise a geometer so much as if the diagonal turned out to be commensurable (Aristotle, *Metaph.*, 983^{a}15)

Introduction

By the time of Aristotle, the mathematical phenomenon of incommensurability was so well known as to be regarded as commonplace.[1] But when it was first discovered, the anomaly of incommensurables posed difficult problems for classical Greek mathematicians.[2] According to a tradition with roots in antiquity, it precipitated a general crisis, some accounts adding the colorful touch that those who first published the result perished shortly thereafter for their indiscretion.[3] While there is no solid evidence to support these claims, there is general agreement among historians that the discovery raised fatal objections to existing mathematical theory. Surviving documents indicate that much effort was put into the project of reformulating mathematics in the wake of this blow, with results that are revolutionary in their scope, and among the highest achievements of that tradition.[4]

*Department of Philosophy, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.
Received 8 *March* 1994; *in revised form* 17 *September* 1994.

[1]Aristotle, *Metaphysics*, 983^{a}15f.

[2]For a recent contrary view see D. H. Fowler, 'The story of the discovery of incommensurability, revisited', in K. Gavroglu, J. Christianidis and E. Nicolaidis (eds), *Trends in the Historiography of Science* (Dordrecht: Kluwer, 1994), p. 221 f.

[3]Scholium to Euclid's *Elements*, Book X, 1. Bulmer-Thomas (ed. and trans), *Selections Illustrating the History of Greek Mathematics* (London: Heinemann, 1941), vol. I, p. 217.

[4]For a persuasive argument that the changes should be regarded as revolutionary see J. Dauben, 'Conceptual revolutions and the history of mathematics: two studies in the growth of knowledge' (1984); reprinted with an appendix in D. Gillies (ed.), *Revolutions in Mathematics* (Oxford: Clarendon Press, 1992), p. 49 f.

Pergamon

Stud. Hist. Phil. Sci., Vol. 26, No. 1, pp. 107–131, 1995

0039–3681/95 $9.50+00.00

The formation and resolution of anomalies like that posed by the incommensurables have a great impact on the development of scientific and mathematical theories. Kuhn's account of scientific revolutions stressed how the accumulation of anomalous data can contribute to the development of new scientific paradigms and the abandonment of old ones. Lakatos showed how counterexamples could lead, by relatively small steps, to more adequate mathematical theories.[5] Along similar lines but in greater detail, Darden presented a set of strategies for anomaly resolution as part of a comprehensive account of theory change in science.[6]

This paper uses Darden's strategies for anomaly resolution to analyze developments in Greek mathematics following the discovery of the incommensurables. We focus on the closely related concepts of ratio and proportion. Our analysis shows that the development of mathematics can have much in common with that of the natural sciences such as genetics, which is Darden's main focus. We first present a brief sketch of Darden's scheme for anomaly resolution, followed by a discussion of the historical background to the discovery of the incommensurables. We then give a schematic summary of the theory of ratio and proportion as it existed before the discovery. The discovery is described, and the subsequent activity is analyzed in terms of Darden's strategies. Concluding remarks assess the adequacy of these strategies for explaining the conceptual changes in the case at hand, and propose a cognitive interpretation of the strategies.

Darden on Anomaly Resolution

In her recent book, Lindley Darden uses a historical analysis of the development of Mendelian genetics to elaborate a general model of theory change in science. This account includes strategies for producing new ideas, for theory assessment, and for changing theories in the face of anomalous data. Her account of anomaly resolution notes that scientists faced with recalcitrant data need not reject their theories entirely, but instead can attempt to save the phenomena by means of changes to an existing theory. For Darden's account to apply, a theory need not be formalized to any great extent, and its elements need not be propositional. Informal reasoning, the use of analogies, of visual or mathematical models, or techniques of data collection and interpretation, all may form part of a theory so understood. The given theory must simply be articulated to some extent; it must contain an indication of what it seeks to explain; and it must be more or less clear how the various components of the theory contribute to the explanations it gives.

[5]Kuhn, *The Structure of Scientific Revolutions* (Chicago: University of Chicago Press, 1962), esp. p. 52 f.; I. Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery*, J. Worrall and E. Zahar (eds) (Cambridge: Cambridge University Press, 1976).

[6]L. Darden, *Theory Change in Science: Strategies from Mendelian Genetics* (New York: Oxford University Press, 1991).

While most theories have some success in explaining a body of data, anomalies turn up with some regularity. These are dealt with in a series of steps.

(1) The anomalous data must be confirmed, or the problem reanalyzed to show that no anomaly exists.

(2) If the existence of an anomaly is confirmed, the source of the mismatch between data and theory must be located. It may be decided that the anomaly falls outside of the data the theory seeks to explain. But if it falls inside, the source of the problem must be sought in one or more components of the theory, and appropriate changes made.

(3) These changes may involve modifications to components, deletion of existing components or the addition of new components. The changes made are subject to a variety of methodological constraints. For instance, the changed theory should provide some understanding of why the anomaly occurs and *ad hoc* explanations should be avoided. Darden describes more specific strategies, naming them in **boldface** type. A component may be **added** or **deleted**, or its domain either **specialized** or **generalized**. A component may be **complicated** or **simplified** by, respectively, adding or deleting conditions. As Darden notes, the specialization of a component's domain is often accomplished by complication as generalization is by simplification.[7] The strategy of **delineate and alter** involves the analysis of a component into several parts and a change to only some of those parts. A **tweak** is a very small alteration which does not fit any of the above categories—for example, changing numerical parameters in a model. Finally, a component may be altered by changing it to a contrary (**propose opposite**).

We can illustrate these strategies by considering the following simple theory.

(a) There are two kinds of bodies: heavenly and terrestrial.

(b) There is only one kind of heavenly body.

(c) All heavenly bodies are weightless.

(d) Heavenly bodies never fall.

(e) All heavy terrestrial bodies, when unimpeded, fall.

Now suppose that, for the first time, a meteor is observed to fall. It is clearly seen at first to be a heavenly body and afterwards just as clearly seen to be a heavy, terrestrial body. In response to this anomalous observation, the theory might be changed as follows.

(a′) There are two kinds of bodies: heavenly and terrestrial.

(b′) There are two kinds of heavenly bodies: stars and meteors; (b′) is obtained from (b) by **proposing an opposite** (i.e. not just one kind of heavenly body).

(c′) Of heavenly bodies, stars are weightless, but meteors are heavy; (c′) is obtained from (c) via the strategy of **delineate and alter**; two cases are

[7]Darden, *op. cit.*, note 6, p. 273.

distinguished, and one is changed (again via **propose opposite**—meteors are heavy instead of weightless).

(d′) Of heavenly bodies, stars never fall; (d′) is obtained from (d) via **specialization**, i.e. the domain (formerly all heavenly bodies) is restricted to stars. This specialization is achieved by **complication** (an additional condition is added).

(e′) All heavy bodies, when unimpeded fall; (e′) is obtained from (e) via **generalization**: the domain is widened from heavy terrestrial bodies to all heavy bodies. This is achieved by dropping the condition "terrestrial", i.e. by **simplification**.

(f′) There is something which usually impedes meteors from falling.

This component is **added**.

Supposing changes to have been made in this way we arrive at the next step.

(4) The changed theory is assessed, using a variety of criteria, including internal consistency, clarity, explanatory and predictive adequacy, scope and generality, extendibility and fruitfulness.[8]

(5) Finally, if all attempts to reform the theory are unsuccessful, one may consider either abandoning the theory or ignoring the anomaly.

We will show that the strategies for component change described in step 3 can be used to help to understand changes in Greek concepts of ratio and proportion.

Classical Greek Concepts of Ratio and Proportion

Early Greek mathematics was, by all accounts, highly dependent upon earlier (Egyptian and possibly Babylonian) sources. Most historians agree, however, that Greek mathematics differed strikingly from other ancient traditions in its strong emphasis on exact results, logical consistency, axiomatics and proof, an emphasis which reached its highest expression in the works of Euclid, Apollonius and Archimedes.

Before the discovery of incommensurability, quantitative mathematics was concerned entirely with whole numbers and their relations. "Number" (*arithmos*) meant a concrete collection of units.[9] Among the relations of numbers, those of "ratio" (*logos*) and "proportion" or "sameness of ratio" (*analogon*) are of particular importance. While several criteria of the equality of ratios were developed, no adequate definition of ratio itself appears to have been formulated. In Book VII of Euclid's *Elements*, the first of the number theoretic books, ratio is not defined at all; in Book V, ratio is given the vague characterization of '... a sort of relation in respect of size between two

[8]Darden, *op. cit.*, note 6, p. 257 f.

[9]J. Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. E. Brann (Cambridge: MIT Press, 1968), p. 46 and *passim*. See, e.g. Euclid, *Elements*, VII, defn. 2: 'A **number** is a multitude composed of units'.

magnitudes of the same kind'. Theon of Smyrna writes 'ratio in the sense of proportion is a sort of relation of two terms to one another, as for example double, triple';[10] Nichomachus has the unhelpful 'a ratio is the relation of two terms to one another'.[11]

We get a better idea of ratio through the concept of "proportion" (*analogon*).[12] *Elements*, VII, definition 20, reads 'Numbers are **proportional** when the first is the same multiple, or the same part, or same parts, of the second that the third is the fourth'.[13] Thus, for instance 2,4, 3,6, are proportional because 2 is the same part of 4 that 3 is of 6, namely one half part. Since proportionality is used interchangeably with "sameness of ratio",[14] we can say that 2 and 4 are in the same ratio as 3 and 6, or 2 is to 4 as 3 is to 6. Each ratio is equal to a ratio whose terms are relatively prime (e.g. 2 is to 4 as 1 is to 2); such terms are also the smallest numbers in the given ratio.[15] It follows that every ratio has a canonical (i.e. lowest terms, relatively prime) form, and this form precisely expresses the multiple/part/parts relationship of any two terms in this ratio. That is to say, given two relatively prime numbers, a,b, all other numbers whose ratio is equal to $a{:}b$ will be of the form na,nb.[16] Ratios thus have names, which can be associated with ordered pairs of relatively prime numbers.[17] The primitive notion of the multiple/part/parts relationship, along with the relation of sameness of ratio (proportionality) thus gives an adequate characterization of ratio, even in the absence of a general definition.[18]

There are other criteria for proportionality. Given any two numbers, the so-called "Euclidean division algorithm" or "reciprocal subtraction" (*anthyphairesis*) can be used to determine their greatest common divisor,[19] and, hence, what multiple, part, or parts one is of the other. This process, which is of considerable importance for our later discussion, may be briefly described as follows: given two unequal numbers, take the smaller and subtract it from the larger as many times as possible; if there is a remainder, take this and subtract

[10]Quoted after T. L. Heath (ed.), *The Thirteen Books of Euclid's Elements*, 2nd edn. (Oxford, 1925; reprinted New York: Dover, 1956), vol. 2, p. 292; cf. J. Dupuis (ed. and trans.), *Les oeuvres de Théon de Smyrne* (Paris, 1892), pp. 118–119.

[11]*Introduction to Arithmetic*, II, 21, 3.

[12]Cf. A. Szabo, *The Beginnings of Greek Mathematics*, trans. A. M. Ungar (Dordrecht: Reidel, 1978), p. 145 f., for a discussion of the origin of the term 'proportional' (*analogon*).

[13]A fourth case, the reciprocal of 'parts', is not included in this definition, but is clearly understood.

[14]Cf. *Elements*, V, defn. 6; also VII, 17, 18, 20, 21, 22. Theon of Smyrna and Iamblichus (both quoted T. L. Heath (ed.), *op. cit.*, note 10, vol. 2, p. 292); also, Aristotle, *Nic. Eth.*, 1131ᵃ31.

[15]*Elements*, VII, 21.

[16]*Elements*, VII, 20.

[17]A fairly thorough list of names—i.e. a classification of ratios—is given by Nichomachus, *Introduction to Arithmetic*, I, 17–23 (summarized in T. L. Heath, *A History of Greek Mathematics* (Oxford, 1921; reprint New York: Dover, 1981), vol. I, p. 101ff.).

[18]In a modern approach, a 'ratio' could be defined as an equivalence class of pairs of numbers under the relation of proportionality. There is no evidence of this type of definition in antiquity; cf. D. H. Fowler, *The Mathematics of Plato's Academy* (Oxford: Clarendon Press, 1987), p. 20.

[19]*Elements*, VII, 1, 2.

it as many times as possible from the smaller of the original two numbers. If there is a second remainder, subtract this successively from the first remainder, and so on. The process (provably) stops after a finite number of steps. Example: let the numbers be 5 and 7:

Step 1, subtract 5 from 7; remainder 2 (once);
Step 2, subtract 2 successively from 5; remainder 1 (twice);
Step 3, subtract 1 successively from 2; no remainder (twice).

Here, 1 is the greatest common divisor (i.e. the numbers are relatively prime): the anthyphairesis can be represented as (once, twice, twice) or [1,2,2]. Note that (10, 14), (15, 21), (20,28), ($5n$, $7n$) all have the same anthyphairesis. Generally, pairs of numbers which are proportional will have the same anthyphairesis, and conversely. Anthyphairesis thus serves as a criterion of proportionality: four numbers are proportional if and only if the first and second have the same anthyphairesis as the third and fourth. A remark of Aristotle suggests that Greek mathematicians may have at one time adopted "having the same anthyphairesis" as the definition of equality of ratio.[20] Of this, more later. Another criterion is based upon "cross-multiplication": the numbers a,b, c,d, are proportional if and only if $ad=bc$.[21]

Some binary operations were defined on ratios. For example, the "compounding" of two ratios corresponds to the modern multiplication of fractions.[22] "Duplicate" and "triplicate" ratio correspond to squared and cubed fractions.[23] These operations are not, however, as is the case in modern presentations, at the center of Greek ratio theory. That is to say, an *arithmetic* of ratios is not prominent. Moreover, it does not even seem to be the case that "compounding" is understood as multiplication.[24]

Much more significant are manipulations of proportions. Among these are **alternation**: from $a{:}b{::}c{:}d$, derive $a{:}c{::}b{:}d$;[25] **inversion**: from $a{:}b{::}c{:}d$ derive $b{:}a{::}d{:}c$;[26] **composition**: from $a{:}b{::}c{:}d$ derive $(a+b){:}b{::}(c+d){:}d$; **separation**: from $a{:}b{::}c{:}d$, with $a>b$ and $c>d$, derive $(a-b){:}b{::}(c-d){:}d$.[27] Further results include: for any numbers a,b,c, $a{:}b{::}ac{:}bc$[28] also, $a{:}c{::}b{:}c$ if and only if $a=b$.[29] It is with

[20] *Topics*, 158^{b}29–159^{a}1; and, for important amplification, the commentary of Alexander of Aphrodisias, *In Topica*, M. Wallies (ed.) (Berlin, 1891), p. 545, quoted in W. Knorr, *The Evolution of the Euclidean Elements* (Dordrecht: D. Reidel, 1975), p. 258: cf. T. L. Heath, *Mathematics in Aristotle* (Oxford: Clarendon Press, 1949), pp. 80 f.

[21] *Elements*, VII, 19.

[22] *Elements*, VIII, 5; cf. VI, defn. 5, 23. In the *Sectio Canonis*, a work on music theory, "compounding" is called "addition" and the inverse operation is called "subtraction"; cf. Szabo, *op. cit*, note 12, p. 137 f.

[23] *Elements*, VIII, 11, 12.

[24] I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (Cambridge, Mass.: MIT Press, 1981), p. 92 f., p. 135 f; cf. Fowler, *op. cit*, note 18, p.138 f.

[25] *Elements*, VII, 13.

[26] Obvious, but not attested in the number theoretic books of the *Elements*; cf. *Elements* V, defn. 13.

[27] Composition is not proved in the arithmetical books, but may easily be along the lines of separation, which is proved in *Elements*, VII, 11.

[28] *Elements*, VII, 17, 18.

[29] Cf. *Elements*. V, 9.

such equivalences of proportions that most of the "algebraic" power of the Greek ratio/proportion concept is lodged.

While it would be an easy matter to define an order relation on ratios on the model of the definition of proportion, no such relation occurs in the number theoretic books of *Elements*. The results pursued in these books do not involve the use of order relations, and it seems safe to assume that they were not prominent in early treatments of ratio theory.

It was apparently assumed that numbers (i.e. whole numbers) and their ratios/proportions could account for all quantitative mathematical relationships, especially in geometry (e.g. in the theory of similar figures), but also in fields such as music theory and astronomy. This assumption entails that all magnitudes can be expressed as integral multiples of some unit. But, given the definition of number as a multitude of units, this meant that all magnitudes could be regarded simply as numbers. There was thus considerable mathematical content—setting aside the mystical aspects[30]—to the Pythagorean dictum that "all is number".[31]

Table 1 gives a summary of the features of the ratio/proportion concept just described.

This concept was put into question, however, by the discovery of the existence of incommensurable magnitudes, magnitudes whose ratio is not equal to any ratio of integers.[32] This discovery was probably made in connection with the problem of doubling the square.[33] It is readily shown that the square built on the diagonal of a given square has twice the area.[34] But the following argument shows that the ratio of side and diagonal cannot be equal to any ratio of integers (or, in anachronistic terms, that $\sqrt{2}$ is irrational).[35]

Suppose AC, the diagonal of a square, to be commensurable with AB, its side.

Let $\alpha{:}\beta$ be their ratio expressed in smallest numbers.

Then $\alpha>\beta$ and therefore necessarily >1.

Now the square on AC is to the square on AB as $\alpha{\cdot}\alpha$ is to $\beta{\cdot}\beta$, i.e.

$$\mathrm{sq}(AC){:}\mathrm{sq}(AB)=\alpha^2{:}\beta^2,$$

and, since $\mathrm{sq}(AC)=2\mathrm{sq}(AB)$ [i.e. the area of the square on the diagonal is twice the area of the given square]

30 On which see Burkert, *Lore and Science in Ancient Pythagoreanism*, trans. E. L. Minar Jr. (Cambridge: Harvard University Press, 1972), p. 465 f.

31 Cf. Aristotle, *Metaphysics*, 986^a 1–2.

32 The discovery is generally dated to the 5th century B.C., with many commentators favoring a date closer to the end of the century.

33 On this, as on many points in the history of ancient mathematics, there is a good deal of speculation and disagreement. We present a traditional view with a good deal of plausibility without claiming to decide the question of how the discovery was first made.

34 Cf. Plato, *Meno*, 82B f.

35 A paraphrase due to T. L. Heath, *op. cit.*, note 10, vol. 3, p. 2; this proof, generally thought to be a later interpolation, occurs in some manuscripts and early editions of Euclid's *Elements* as X, 117.

Table 1 Summary of ratio and proportion theory before the discovery of incommensurables

1. Number and fundamental operations
(a) Definition of number: a number is a multitude composed of units.
(b) Definition of addition: the formation of a new multitude of units from two given multitudes (union of disjoint sets).
(c) Definition of multiplication: repeated addition.
2. Ratio and proportion
(a) Characterization of ratio: a multiple/part(s) relationship between numbers.
(b) Definition of "same ratio" (proportionality): being in the same multiple/part(s) relationship.
(c) Every ratio has a unique lowest terms representation, which can serve as a name.
3. Equivalent criteria of equality of ratios
(a) $a:b::c:d$ if and only if $ad=bc$.
(b) $a:b::c:d$ if and only if (a,b) have the same anthyphairesis as (c,d).
(c) $a:b::c:d$ if and only if both have the same lowest terms representation.
4. Relations/operations on ratios/proportions
(a) Order relation $a:b < c:d$ iff $ad< bc$ (not prominent).
(b) Compounding ratios: the composition of $a:b$ and $c:d$ is $ac:bd$.
(c) Various operations on proportions, e.g. alternation, inversion, separation, conversion.
5. Domain of application of ratio concept
All magnitudes, geometrical objects in particular.
6. Applications: further assumptions
(a) In geometry, all quantitative relationships may be represented by (whole) number ratios.
(b) Equivalently, all geometrical magnitudes can be expressed as numbers (i.e. integral multiples of some unit).

$$\alpha^2=2\beta^2.$$

Therefore, α^2 is even, and, therefore, α is even.
Since $\alpha:\beta$ is in its lowest terms, it follows that β must be *odd.*
Put $\alpha=2\gamma$;
therefore, $4\gamma^2=2\beta^2$,
or $\beta^2=2\gamma^2$,
so that β^2, and therefore β, must be *even.*
But β was also odd:
which is impossible (Fig. 1).

It follows that if the side is measured by some unit, the diagonal cannot be, and *vice versa*. In short, it is impossible to determine a unit which will measure both of the magnitudes, and they cannot therefore both be numbers. The assumption that whole numbers and their ratios are sufficient for mathematics is thus shown to be false in one of the simplest of geometric constructions. We now consider the responses to this discovery in terms of Darden's five steps in anomaly resolution: confirmation, localization, theory change, assessment, and possible abandonment.

Fig. 1. Doubling the square; the incommensurability of side and diagonal.

Anomaly Resolution and Conceptual Change

Confirmation of the Anomaly

Since the anomaly of incommensurability entailed such important consequences for existing mathematics, it is likely that the proofs were checked and rechecked. If any doubt remained about the significance of this example, it was dispelled by the later discovery, often attributed to Theodorus (*c.* -430), of results equivalent to the irrationality of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, $\sqrt{15}$, $\sqrt{17}$; and by the general result, often attributed to Theaetetus (*c.* -390) equivalent to the irrationality of $\sqrt{n}$ whenever n is not a perfect square.[36] There could thus be no question of the incommensurability of the side and diagonal of a square being an isolated result. The existence of the anomaly was thus abundantly confirmed, as required by Darden's step 1.

Localization of the Anomaly

The problem posed by the incommensurables involves elements of geometric construction, logical structure, number theory, and ratio/proportion theory. The source of the problem might, therefore, be sought in any of these areas. In particular, one might adopt any of the following tactics:

Tactic 1, limit geometrical construction postulates in such a way that incommensurable magnitudes could not occur;

Tactic 2, ignore the inconsistency and remain satisfied with approximations to the ratios of incommensurables;

Tactic 3, generalize the concept of number;

Tactic 4, limit the application of ratio in geometry and reformulate as much geometry as possible without the use of ratio; or

Tactic 5, generalize the concepts of ratio and proportion.

The first two tactics do not appear to have had any currency in classical Greek mathematics. Construction by means of straight-edge and compass was doubtless considered to be too well founded to be questioned in any significant way. The option of ignoring inconsistencies and accepting approximations in the place of exact results was also quite unlikely. While the other mathematical

[36]Cf. Plato, *Theaetetus*, 147D–148B; and an anonymous scholium to *Elements*, X, 9, *Euclidis Opera Omnia*, Heiberg (ed.) (Leipzig, 1883–1916), vol. V, p. 450, line 16 ff; cf. Szabo, *op. cit*, note 12, p. 76 f.

traditions of antiquity might well have regarded the discovery as interesting but relatively unimportant, the Greeks, sensitive as they were to logical foundations, could not accept inconsistencies.[37] To see why this is so, we need only focus for a moment on the role of proof by reduction to absurdity in Greek mathematics. The logical basis of these arguments is the observation that one derived contradiction is sufficient to show the untenability of an entire system of assumptions. Inconsistencies cannot, in this approach, be isolated and ignored, but rather force the abandonment of the entire system. There is little doubt that the foundational approach was an important element of the phenomenon of incommensurability, being recognized, not as a curiosity, but as a *problem* which demanded a reformulation of significant portions of mathematics.

Tactic 3: Generalize the Concept of Number

Nor did tactic 3 have any popularity, as the concept of number (*arithmos*) also remained constant in the various responses to the incommensurables. From the point of view of many modern accounts of Greek mathematics, this seems to constitute a paradox, since the discovery of incommensurables in geometry is often described as the discovery of irrational numbers. Nothing could be farther from the thought of Greek mathematicians, however. As Heinrich Scholz remarked, this question is profoundly misguided since the Greeks did not even construct a theory of *rational* numbers.[38]

Why was this so? Greek mathematics of the classical period, most probably under the influence of philosophy, was conceived as a foundational system. The goal was to establish incontrovertible axioms and methods of inference. The notion of building uninterpreted, purely "formal" systems is not in obvious harmony with this goal. But, lacking a sophisticated conceptual apparatus that was developed only many centuries later, the only way apparent to the Greeks to generalize the concept of number would be to admit such systems.

Consider the Greek concept of number. As mentioned before, a number is a multitude of units. Units are indivisible individuals. That is to say, we should think of a typical unit not as a geometrical magnitude like a line, but instead as an individual like, say, a sheep. In order to have a full arithmetic of rational *numbers*, it would be necessary to be able to divide units at will, since (for example) 1/2 would be a rational number produced by dividing one into two

[37] Cf. Knorr, *op. cit.*, note 20, p. 4. Depending on the dating of the discovery and the transmission of results, however, there may have been cases where the foundational problems were simply ignored. Hippocrates of Chios (*c.* −450), for example, in his quadratures of lunes (cf. I. Bulmer-Thomas, *op. cit.*, note 3, vol. I, pp. 234 ff.), freely uses proportion theory. It is possible that he was working after the discovery of incommensurables and before the generalization of the ratio concept. If so, his work would be based upon inadequate foundations.

[38] 'Warum haben die Griechen die Irrationalzahlen nicht aufgebaut?' *Kant-Studien* **33** (1928), 35–72.

parts. But if one is divisible, it could not have been a unit in the first place. This observation is very concrete—there is no such thing as half of a sheep. To speak of dividing one, in this way of thinking, is simply an elaborate metaphor for starting with some even number of sheep in the first place, and calling this even number, by an abuse of language, *one*. That is, if in performing an arithmetical calculation one divides the unit, by seven say, one has simply misformulated the problem: instead of this alleged unit, the problem assumes another unit, seven of which make up the original "unit". Thus there is no need to develop an uninterpreted arithmetic of fractions; by normalizing at every step, the procedures can one and all be treated as operations on natural numbers. Thus Plato writes:

> You know how steadily the masters of [arithmetic] repel and ridicule any one who attempts to divide absolute unity when he is calculating, and if you divide, they multiply, taking care that one shall continue one and not become lost in fractions.[39]

One can see how tedious this practice might become if, instead of simply normalizing only once, at the end of a calculation, one were forced to do so at every step of the way. In this sense, the foundational impulse would be an impediment and an irritant. Nonetheless, the point is quite clear: there is no need, for the purposes of ordinary arithmetic (and in the absence of infinite processes), to speak of rational *numbers* in addition to ratios of natural numbers.

This being said, it is understandable that the Greeks did not develop a theory of rational *numbers*. Instead, they developed a set of techniques for manipulating ratios and proportions of numbers. And when the discovery of incommensurables occurred, it was interpreted within this framework, rather than as showing the need for irrational numbers. Finally, it should be noted that the techniques of proportion theory were quite sophisticated and flexible. With them, for instance, Archimedes was able to accomplish operations equivalent to the summing of infinite series, to determine surface areas, volumes, arc lengths, etc—i.e. to solve problems now generally treated by means of real number theory and the calculus. So it should not be thought that the lack of a theory of real numbers was a dead weight on Greek mathematics; the theory of proportion was powerful enough to do many of the things a theory of real numbers can do.

While geometrical construction, logical consistency, and the concept of number appear to have been unquestioned, the other two tactics were thoroughly investigated. There is evidence of attempts to reformulate geometric results without the use of proportion, as well as two attempts to generalize the ratio concept.

[39] *Republic*, 525D. See also B. L. van der Waerden, *Science Awakening*, trans. A. Dresden (New York: Science Editions, 1963), p. 115.

Tactic 4: Geometry Without Ratio

Since the unlimited application of ratio to geometry results in absurdities, by limiting this application one might avoid these difficulties. However, in so doing, one loses a good deal of mathematics, and the question immediately arises, how much of the lost material can be reconstructed without the help of ratios? There is evidence of considerable effort in this direction.

The first four books of Euclid's *Elements*, for example, develop a large body of geometric results without the use of ratio or proportion. Commentators have noticed that many of the results of this part of *Elements*, above all those of Book II, deal with subjects which would normally be treated in terms of equations, numbers, or proportion theory.[40] It is very likely that this geometric approach to what are *prima facie* algebraic results was adopted in order to avoid the problems of incommensurables. Because "number" was restricted to whole numbers, and these could not express all geometric magnitudes, theorems about numbers would not have the generality required by geometry. For the purposes of geometry such results had to be cast in a more general form. This was done by substituting geometric constructions for what previously had been accomplished through operations on numbers. To take one example, consider *Elements* II,1:

> If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

Before the discovery of the incommensurables, this result may well have been treated as simultaneously concerning numbers and geometric figures. [Algebraically, the result can be expressed in the equation $M(A+B+C+...)= MA+MB+MC+....$]. For if all magnitudes were commensurable, each segment in the diagram could be represented by a finite number of dots (Fig. 2). A rectangle would represent simultaneously a multiplication of two numbers and a geometrical construction, and the geometrical theorem would be an instance of the arithmetical result. The discovery of incommensurability renders this approach invalid. Since not all magnitudes can be measured by numbers, the geometric theorem has greater generality; the arithmetic version has only limited (and generally uncertain) application in geometry.

Instead of relying on arithmetic, then, this approach replaces arithmetical procedures by geometric constructions. Addition had previously been defined for numbers only. After the discovery of incommensurables, it is reinterpreted in separate cases for numbers, lines, planes and solids. Two numbers are added as before; to add two lines, one extends one of the lines by a length equal to the other; to add two areas, one maps them on to rectangles having a common

[40]For example T. L. Heath, *op. cit.*, note 17, vol. I., p. 150 f.; B. L. van der Waerden, *op. cit.*, note 39, p. 118 f., p. 125 f.

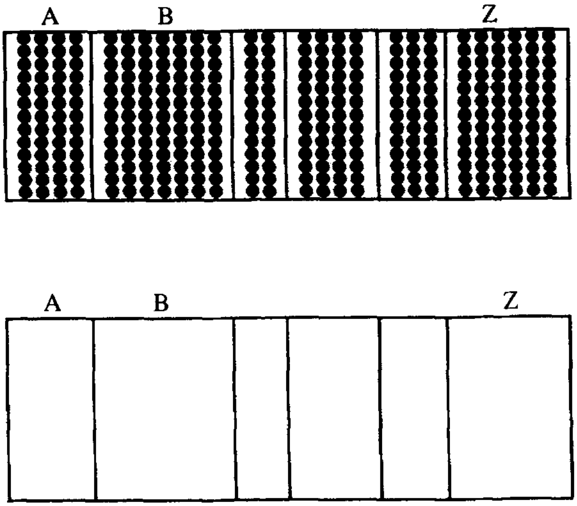

Fig. 2. The result M(A+B+C+...)=MA+MB+MC+... *in the arithmetical theory and geometrical algebra.*

base, and juxtaposes them, and similarly for solids.[41] Multiplication is somewhat more complex. As a binary operation admitting heterogeneous inputs, it occurs in a variety of forms: (number, number)→number, (number, line)→line, (number, plane)→plane, (number, solid)→solid, (line, line)→plane and (line, plane)→solid. For two numbers, multiplication is as before. For a number and a magnitude (line, plane or solid), multiplication is repeated addition, with addition defined as above. For two lines, multiplication is interpreted as the construction of a rectangle: for a line and a plane, it involves the construction of a rectangular solid. Multiplication is not possible for any other combinations of magnitudes. This is so because rectangular solids have no higher dimensional analogs in Greek geometry. Subtraction and division, where possible, are interpreted in the obvious way as inverses of addition and multiplication. Finally, square roots (geometric means) can be interpreted for two lines as the line resulting from a construction in a semicircle with two given lines.[42] As was the case with multiplication, dimensional restrictions place limits on this interpretation.

In terms of Darden's strategies, this approach **specializes** the domain of ratio and proportion to numbers by excluding general geometrical magnitudes. In order to recover mathematical content, it reinterprets fundamental operations, following the strategy of **delineate and alter**, splitting the notions of addition, subtraction, multiplication, etc. into several different operations on different domains, while maintaining the original concept for the domain of numbers.

[41]We omit the mention of times and other non-geometrical magnitudes. Book two deals only with planes, but the results are readily generalizable.

[42]For details, see *Elements*, II, 14; cf. *Elements*, VI, 13.

The important Pythagorean components concerning the applicability of numbers and their ratios to all quantities are **deleted**.

While this approach was to a large degree successful in avoiding the problems of incommensurability, it had costs. First of all, there is no evidence that this approach ever succeeded in reformulating the results of the theory of similar figures (i.e. there was a significant loss of content). Second, there are difficulties in the interpretation of multiplication. Because multiplication is interpreted as the construction of a rectangle from given line segments, there is a change of dimension involved with the operation. Since Greek geometry allowed only three dimensions, the theory puts a restriction on "multiplication" which does not obtain for the multiplication of numbers.[43] Similar problems may arise with the construction of geometric means. The loss of expressive power is also considerable; instead of having, as before, quantitative means for speaking of and proving many geometric results, one must avoid these wherever incommensurability threatens to undermine the logical basis of the argument. This often results in lengthy and inelegant paraphrases of results which might be better expressed in quantitative terms. Nevertheless, even though this strategy fell short of a satisfactory solution, it provided a number of results (the generalization of the notion of magnitude and the operations of addition, multiplication, etc.) which were maintained throughout later Greek mathematics.

Tactic 5: Generalization of the Ratio Concept

The fifth strategy, that of extending the concepts of ratio and proportion, was successfully pursued at least once and possibly twice. We have documented evidence of one such attempt, preserved as Book V of Euclid's *Elements* and usually attributed to Eudoxus of Cnidos (*c.* -370). Some historians have argued that there was, in addition, an earlier attempt, probably due to Theaetetus, based on the Euclidean division algorithm (*anthyphairesis*).[44] Without attempting to decide this question, we will present a sketch of this alleged earlier theory (tactic 5a) as well as of that of *Elements*, Book V (tactic

[43]See, for example, Pappus, *Collection*, VII, 38-40, trans. I. Bulmer-Thomas, *op. cit.*, note 3, vol. II, p. 601: 'If there be more than six straight lines, it is no longer permissible to say "if the ratio be given between some figure contained by the remainder", since no figure can be contained in more than three dimensions. It is true that some recent writers have agreed among themselves to use such expressions, but they have no clear meaning...'. Pappus goes on to suggest a way of dealing with the difficulty by means of composition of ratios (*ibid*).

[44]For example O. Becker, 'Eudoxos-Studien I. Eine voreuklidische Proportionlehre und ihre Spuren bei Aristoteles und Euklid', *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abt. **B 2** (1933), 311–333; B. L. van der Waerden, *op. cit.*, note 39, p. 175 f.; W. R. Knorr, *op. cit.*, note 20; D. H. Fowler, 'Ratio in Early Greek Mathematics', *Bulletin of the American Mathematical Society*, New Series **1** (1979), 807–846; D. H. Fowler, 'Anthyphairetic ratio and Eudoxan proportion', *Archive for History of Exact Sciences* **24** (1981), 69–72; D. H. Fowler, *op. cit.*, note 18; A. Thorup. 'A Pre-Euclidean Theory of Proportions', *Archive for History of Exact Sciences* **45** (1992), 1–16.

5b). A third opportunity, that of generalizing the cross-multiplication criterion of proportionality (i.e. $a:b::c:d$ iff $ad=bc$), was apparently not pursued. This may have been for technical reasons: in order to use the criterion, multiplication would always have to be possible; but, as noted above, because Greek geometry used only three dimensions, multiplication on the model of producing an area from two line segments is not always possible (e.g. an area with an area). One way to get around this is to map all quantities onto line segments via the specification of a unit (with a tacit completeness axiom); but this approach was not systematically exploited until the sixteenth century (most influentially in Descartes' *Geometry* of 1637), and was there dependent upon (or at least greatly aided by) advances in algebra unavailable to the Greeks.

Tactic 5a: The Anthyphairetic Definition of Ratio

The Euclidean division algorithm, or *anthyphairesis*, could, as noted above, be used as a criterion of equality of ratio (proportionality) of two pairs of integers. The process of anthyphairesis is also interpretable geometrically. Given two line segments, for example, one may repeatedly subtract the smaller from the greater, and then subtract the remainder from the smaller of the original lines, and so on—similarly for areas and volumes. While for all cases with numbers the division algorithm terminates after a finite number of steps, this is in general false for geometrical magnitudes, as is made clear by some rather striking constructions.[45] Nonetheless, there is a sense in which the algorithm approaches a limit, namely that the remainders approach zero as the number of steps goes to infinity (a result which follows from *Elements*, X, 1). The anthyphairesis, moreover, provides convergent sequences of upper and lower bounds with a common limit equal to the (as yet undefined) ratio of the two magnitudes, essentially by producing the continued fraction expansion of the ratio.[46] It is easy to show that: (1) when the anthyphairesis of two magnitudes is finite, they are commensurable (the anthyphairesis in this case constructing the common measure); and (2) that when it is infinite, they are incommensurable (*Elements*, X, 2). Since having the same (finite) anthyphairesis is a necessary and sufficient condition for the equality of two ratios, the possibility suggests itself of taking this criterion as a definition, and extending it to those cases where the algorithm is non-terminating. Aristotle's remark that "having the same antanairesis" is the definition of having the same ratio[47] may thus point to an attempt to extend the ratio/proportion concept to cover the case of incommensurables. In what follows, we assume that this was so.

[45]For amplification, see Fowler, *op. cit.*, note 18, p. 33 f., p. 305 f.

[46]For details, see Fowler, *op. cit.*, note 18, p. 309 f.

[47]*Topics*, $158^{b}29$-$159^{a}1$; according to Alexander (of Aphrodisias, Aristotle's 'antanairesis' is synonymous with 'anthyphairesis'; cf. note 20 above.

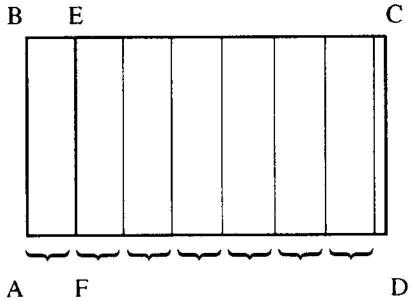

Fig. 3. ABEF:FECD::AF:FD.*The anthyphairesis of the line induces an equivalent anthyphairesis of the rectangle.*

As noted above, the results of "geometry without ratio" or geometric algebra are maintained. In particular, the requirements of homogeneity are upheld. Numbers can only be compared with other numbers, lines with other lines, and similarly for planes, solids and times. This is so because a line, no matter how many times it is multiplied, will never exceed any area, however small; the anthyphairesis of a line and a plane is therefore not possible. Instead, pairs of lines are compared with pairs of solids, or times, etc., and proportion is interpreted as a relation between two pairs of homogeneous magnitudes.

With this new definition, some results are obvious. Consider, for example the rectangle *ABCD*, cut by a line *EF* parallel to the sides *AB*, *CD* (Fig. 3). The anthyphairesis of the rectangles *ABEF* and *FECD*, constructed by first subtracting the smaller of the two from the greater, induces an anthyphairesis of the lines *AF*, *FD* which is clearly the same; the ratios are therefore equal, a result which corresponds to the earlier theorem for numbers: $a{:}b{::}ac{:}bc$.[48] Several other propositions are also easily obtained, among them the **inversion** of proportions: $A{:}B{::}C{:}D$ if and only if $B{:}A{::}D{:}C$; **composition** $A{:}B{::}C{:}D$ yields $(A+B){:}B{::}(C+D){:}D$; and **separation**: $A{:}B{::}C{:}D$ yields $(A-B){:}B{::}(C-D){:}D$.[49] **Alternation** (i.e. for four homogeneous magnitudes A,B,C,D, $A{:}B{::}C{:}D$ if and only if $A{:}C{::}B{:}D$), however, may have been more difficult to prove. A statement made by Aristotle has been interpreted by a number of commentators to mean that the proof of alternation based on the anthyphairetic definition of proportion treated the cases of lines, planes, solids and times separately.[50] The result $A{:}B{::}C{:}D$ if and only if $AD=BC$ may also have been proved separately for the cases where: a) A,B,C,D are lines and AD,BC rectangles, and (b) A,B are lines, C,D rectangles, and AD,BC rectangular solids.[51] Finally $A{:}B{::}A{:}C$ if and

[48]This is the result referred to by Aristotle in *Topics*, 158^b29–159^a1; see also Knorr, *op. cit.*, note 20, p. 334 f., p. 263.

[49]Thorup, *op. cit.*, note 44, p. 13, p. 9; also van der Waerden, *op. cit.*, note 39, p. 177.

[50]*Posterior Analytics*, 74^a17 f. ; for discussion of this remark, see Knorr, *op. cit.*, note 20, p. 264; van der Waerden, *op. cit.*, note 39, p. 177 f.; also Thorup, *op. cit.*, note 44, p. 10 f., who suggests that a general proof was possible within the framework of Greek mathematics.

[51]Knorr, *op. cit.*, note 20, p. 262 f; 338 f.

only if $B = C$ is a result which may well have been very difficult to establish with the anthyphairetic definition.[52]

For numbers, **compounding** ratios (corresponding to the multiplication of fractions), although not prominent, was relatively straightforward. For anthyphairetic ratios this is far from the case, as Fowler has described in some detail.[53]

An order relation is also easily definable with the anthyphairetic definition.[54] If, however, as Knorr has suggested, this definition was developed by Theaetetus for the purposes of investigating incommensurable magnitudes, it seems likely that order relations were not prominent in the theory. The parts of the *Elements* often attributed to Theaetetus, Books X and XIII,[55] do not make any significant use of inequalities, in marked contrast to Book XII (discussed below). This is not to say that inequalities as such had no use, however. Anthyphairesis, as noted above, gives, at successive steps, upper and lower bounds for a given ratio. There is some evidence to suggest that this fact was exploited to obtain approximations for ratios.[56] We merely suggest that order relations among ratios may not have been of central importance in the development of the theory.

A final point deserves mentioning. We remarked above that no satisfactory definition of arithmetical ratio is to be found in surviving documents, but that instead the notion of ratio was grasped via an enumeration of individual integral parts/multiple relationships. Similarly, the notion of quantity could be grasped, not by a general definition, but by listing individual numbers. Just so for the case of the anthyphairetic definition of ratio, which tells us what it means for two ratios to be equal, but not what ratio itself is. Moreover, the discovery of incommensurables shows that the older approach for naming ratios is inadequate, because many relationships between magnitudes will not occur in this list, and many magnitudes will not be expressible as numbers. Since there was no general definition of ratio or quantity, the discovery of incommensurables puts into question whether one could have an adequate grasp of these notions at all. The framing of the discovery suggests a way out, however: $\sqrt{2}$ is constructed as the side of a square which has twice the area of a given square. Although not expressible as an integral multiple of any aliquot part of the side, the quantity can still be represented as "the side of the doubled square". Similarly for $\sqrt{n}$, whenever n is not a perfect square. Book X, perhaps the most accomplished of the *Elements*, goes beyond such straightforward results to investigate and classify more complicated irrationals: those of the

[52]Knorr, *op. cit.*, note 20, p. 340; for a contrary view, see Thorup, *op. cit.*, note 44, p. 2, p.10.

[53]Fowler, *op. cit.*, note 18, p. 115.

[54]See Knorr, *op. cit.*, note 20, p. 334, defn. 4; Thorup, *op. cit.*, note 44, p. 13, defn. 3.

[55]For example by van der Waerden, *op. cit.*, note 39, p. 173 f.; Knorr. *op. cit.*, note 20, p. 301; Heath (ed.), *op. cit.*, note 10, vol. III, p. 2 f.; p. 438 f.

[56]For classical references and helpful commentary see Fowler, *op. cit.*, note 18, p. 51 f.

form $\sqrt{\sqrt{a} \pm \sqrt{b}}$ where a, b are line segments commensurable with a given unit. There are indications that this approach was at least partially extended to cubic irrationals as well.[57] By thus extending the expressive apparatus for magnitudes and thus for quantity relationships these results begin to address the difficulty raised by the incommensurables.

In terms of Darden's strategies, the redefinition of ratio (tactic 5a) involves, to begin with, all the changes described above for tactic 4, geometry without ratio. The notion of magnitude is **generalized** from (discrete) numbers to various types of continuous magnitudes (e.g. lines, planes, solids, times). This generalization is accomplished via **simplification**, as former constraints on magnitudes (i.e. those properties which are specific to integers) are dropped. Consequent upon these changes, and the changes to the concepts of "addition" and "multiplication", ratio is **generalized** via **simplification** to pairs of homogeneous magnitudes, and proportion to pairs of such pairs. Some results require additional constraints: for example, the alternation of proportion requires that all four magnitudes be of the same kind (**complication**). Given the geometric interpretation of anthyphairesis, results of ratio theory which were entirely general in the older theory have different meaning depending upon the magnitudes involved—an instance of **delineate and alter**. This is particularly prominent in the case of the alternation of proportion and the result $A{:}B{::}C{:}D$ if and only if $AD=BC$, where separate proofs may have been given for different types of magnitudes. Finally, the Book X classification of irrationals and their ratios, extending the expressive means for describing ratios, involves the **addition** of new components.

The reformulation of the definition of proportion requires separate comment. Assuming that the original definition was in terms of parts and multiples, the transition to the anthyphairetic definition presupposes first the **proof of equivalence** of the anthyphairetic criterion of proportionality with the parts/multiple definition (i.e. on the domain of numbers); the **substitution of equivalents for equivalents** (i.e "having the same anthyphairesis" for "having the same part/multiple relationship"), a geometrical reinterpretation of anthyphairesis along the lines of geometry without ratio (an instance of **delineate and alter**) and, finally, the **generalization** to the case where anthyphairesis is non-terminating. The first two of these changes—**establish equivalence**, **substitute equivalents for equivalents**—do not, to our knowledge, appear in Darden's account. We would like to underline the importance of these strategies for the development of mathematics. Consider one further example: a number of nineteenth-century studies developed theorems concerning open and closed subsets of the real line. On this basis, a number of propositions could be proved which had reference only to open sets and their properties. While equivalent for

[57]Knorr, *op. cit.*, note 20, p. 270.

a range of results to the characterization of the real line in terms of its point-subsets, the former understanding generalizes in ways which the latter does not—giving rise to *topology* as a branch of mathematics independent of and prior to the study of the real line.

The anthyphairetic theory of ratio is largely successful: it provides a definition of proportion which applies to both commensurable and incommensurable magnitudes. Under suitable reinterpretations, the main results of proportion theory are also saved. As was the case with "geometry without ratio", these reinterpretations are not without their costs. In particular, the theorems of anthyphairetic ratio theory have concrete and specific geometrical meaning—for instance, the inversion of proportion concerns not ratio as such, but pairs of lines, planes, solids, times. Where the Pythagorean number concept applied to all quantities without exception, the ratio concept which emerges from anthyphairesis is a patchwork composed of different geometrical results. Instead of propositions which apply to all ratios, the theory provides separate theorems (with similar statements) for each type of magnitude. Aristotle's remarks in the *Posterior Analytics* suggest that, formal considerations aside, this disparity was recognized as a defect by contemporary mathematics.[58] Such objections, however, do not have obvious or direct bearing on the work of practicing mathematicians, for whom the anthyphairetic theory is eminently serviceable.

Given this general adequacy, historians who hold that the anthyphairetic definition was once current are faced with the difficulty of explaining why a different theory, that of Eudoxus (*Elements*, Book V), became the received view. A number of explanations have been offered, some of them centering on the technical development of the anthyphairetic theory,[59] others pointing to differences in cognitive style between Theaetetus and Eudoxus, the alleged creators of the two theories.[60] To this latter suggestion we add the following: the problems investigated by these two mathematicians were quite different. Theaetetus was concerned primarily with the classification of constructible incommensurables, and with the construction of the regular solids. Eudoxus, on the other hand, is known for the method of exhaustion, where an ingenious use of inequalities allows the determination of ratios not within the direct grasp of classical methods. If, as Knorr has suggested, the anthyphairetic theory was a special purpose theory intended to deal with the problems of Books X and XIII, it seems reasonable that a different set of problems could lead to a different attack on the problem of generalizing the ratio concept—this all the more so if the anthyphairetic theory was not worked out in great detail.

[58] 75^{a}17 f.
[59] Knorr, *op. cit.*, note 20, p. 265 f., p. 338 f. Thorup, *op. cit.*, note 44, p. 3.
[60] Van der Waerden, *op. cit.*, note 39, p. 189.

Tactic 5b: Generalization of Ratio—Eudoxus' Ratio Theory

While the anthyphairetic ratio theory just described is an almost entirely speculative reconstruction, Book V of Euclid's *Elements* preserves a fully developed general ratio theory which is applicable to commensurable and incommensurable magnitudes alike. Whereas in the older theory ratios were binary relations between numbers, in Eudoxus' theory ratio is a binary relation between magnitudes "of the same kind" which can be "multiplied to exceed one another"; and proportion is a binary relation between two pairs of such magnitudes. A notable feature of this later theory is its increased abstractness. Where the arithmetical theory of proportion applied only to (concrete) numbers, and the anthyphairetic theory was (presumably) rooted in geometrical constructions, the theory of Book V is conceived so as to apply to all magnitudes which meet a certain condition: the so-called axiom of Archimedes: 'Magnitudes are said to **have a ratio** to one another which are capable, when multiplied, of exceeding one another'.[61] Although readily interpretable in geometric terms, the definitions and propositions of Book V make no essential reference to geometric objects. Proofs involve only the properties postulated to belong to general magnitudes. This is not to say that geometry is irrelevant: the main purpose of the theory is clearly to support geometric applications, notably to the theory of similar figures and the method of exhaustion. Geometrical magnitudes, moreover, provide instantiations of the primitive terms of the theory; and a limited completeness principle for geometry is grounded not on a postulate, but on a geometric construction.[62]

As noted above, "ratio" itself is not rigorously defined, but only vaguely characterized as "a sort of relation in respect of size". Instead, equality of ratio ("being in the same ratio") is defined. The basis of the definition of equality of ratio is the comparison of different (integral) multiples of magnitudes. As is the case with the anthyphairetic definition, magnitudes must be of the same kind (e.g. one can compare a time with a time, a volume with a volume, an area with an area, a length with a length, but not a time with a length or an area with a volume).

The definition of "same ratio" is as follows (*Elements*, V, defn. 5):

[61] *Elements*, V, defn. 4; cf. Archimedes, *On the Sphere and Cylinder*, Book I, postulate 5, in P. Ver Eecke (ed. and trans.), *Les Oeuvres complètes d'Archimède, suivies des commentaires d'Eutocius d'Ascalon* (Paris, 1960), vol. I, p. 6.

[62] The existence of the 'fourth proportional' (i.e. given any three magnitudes A,B,C, there exists a magnitude D such that $A{:}B{::}C{:}D$) is proved constructively for lines at *Elements*, VI, 12. Aristotle (*Physics*, 207^b30f) expresses the existence of a fourth proportional directly as an axiom of geometry. J. Itard, *Les livres arithmétiques d'Euclide* (Paris: Hermann, 1961), p. 57n, notes that the existence of a fourth proportional is used only once, and there tacitly, in Book V. It is possible that a completeness axiom was not included in ratio theory for methodological reasons: for instance, in order that the theory would also apply to magnitudes which do not satisfy completeness, such as numbers.

Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

"Equimultiples" here means equal integral multiples (which can be, in general, constructed for geometrical magnitudes). The definition thus relies on a comparison of all possible pairs of such multiples; that is, in order to determine whether A and B are in the same ratio as C and D (where A,B are magnitudes of the same kind, and C,D are also magnitudes of the same kind), all possible integral multiples nA, mB must be compared with mC, nD. The ratios are equal if and only if:

(1) whenever $mA < nB$, $mC < nD$;
(2) whenever $mA=nB$, $mC=nD$;
(3) whenever $mA > nB$, $mC > nD$.

Many of the results of the arithmetical proportion theory are also reformulated more generally in Book V. Alternation ($A:B::C:D$ if and only if $B:A::D:C$), composition ($A:B::C:D$ implies $(A+B):B::(C+D):D$), separation ($A:B::C:D$ implies $(A-B):B::(C-D):D$) are given simple and elegant general proofs.[63] Similarly for the propositions: $A:C::B:C$ if and only if $A=B$, and $C:A::C:B$ if and only if $B=C$.[64] Other results of arithmetical proportion theory are treated geometrically, outside the scope of Book V. For example, a version of the theorem: $a:b::ac:bc$ is proved by construction in the case where a, b, c, are lines.[65] Similarly, $a:b::c:d$ if and only if $ad=bc$ is proved for the case where a, b, c, d, are lines and ad, bc rectangles.[66]

"Compounding" of ratios makes an appearance for general ratios without, however, being adequately defined.[67] It is possible that the notion of "compounding" ratios was a survival from an earlier version of the theory of similar figures based on the arithmetical theory of proportion, or that it is a later addition, influenced perhaps by contact with Babylonian methods of calculation.[68] In any case, the operation is not at all prominent. Duplicate and triplicate ratio, however, are given satisfactory definitions.[69] The definitions are extensional rather than operational, i.e. when $A:B::B:C$, $A:C$ is said to be the duplicate ratio of $A:B$. This notion is readily generalizable. Archimedes speaks of the one-and-a-half ratio[70] (i.e. corresponding to $(A:B)^{3/2}$), and it seems quite

[63] *Elements*, V, 16, 18, 17.
[64] *Elements*, V, 7,9.
[65] *Elements*, VI, 1.
[66] *Elements*, VI, 16: for generalizations to parallelograms and solids, see VI, 14 and XI, 25.
[67] See *Elements*, VI, defn. 5 (omitted in most editions), and VI, 23.
[68] Itard, *op. cit.*, note 62, pp. 62–63.
[69] *Elements*, V, defn. 9, 10.
[70] *On the Sphere and Cylinder*, Book II, prop. 8, Ver Eecke (ed.), *op. cit*, note 61, vol. I, p. 114.

possible that he understood something like positive rational exponentiation of ratios in its full generality.

Apart from the theory of proportion, one of the most important developments of Book V is the definition of an order relation (greater ratio) along with the statement and proof of a number of results paralleling the theorems of proportionality, for example: $A>B$ entails $A:C>B:C$;[71] $A:B::C:D$ and $A \gtreqless C$ entails $B \gtreqless D$.[72]

The importance of these and related theorems becomes clear in Book XII where the method of exhaustion is used to establish ratios between figures by means of an ingenious manipulation of recursively specified inequalities. The results are established indirectly, by showing the absurdity of a ratio being, respectively, greater than or less than another ratio. Using these techniques, a number of important theorems are proved: for example, circles are in the same ratio as the squares constructed on their diameters;[73] the ratio of a cone to the cylinder with the same base and height is 1:3;[74] and spheres are in the triplicate ratio of their diameters.[75] These techniques were further exploited by Archimedes with remarkable success.

In terms of Darden's strategies, the ratio theory of Book V **generalizes** the concept of magnitude from numbers (in the arithmetical theory) and geometrical magnitudes (in the anthyphairetic theory) to systems satisfying several (explicitly or implicitly stated) criteria. This generalization is achieved via **simplification**, as the properties of numbers which are not properties of general magnitudes are dropped. As was the case with the anthyphairetic theory, the ratio theory of Book V presupposes, for its geometrical applications, the reinterpretations of addition, multiplication, and so on developed in "geometry without ratio". The ratio concept is **generalized** from a relation between two numbers (or two homogeneous geometrical magnitudes) to one between two magnitudes satisfying the archimedean axiom. This, again, is achieved via **simplification** of the constraints on magnitudes. Proportion, similarly, is **generalized** from four numbers to two pairs of homogeneous magnitudes, as are some of the results of proportion theory. For example, inversion, composition, and separation of proportions are generalized from the case of four numbers to that of two pairs of homogeneous magnitudes. Alternation generalizes in this respect, but adds a constraint (**complication**): $A:B::C:D$ if and only if $A:C::B:D$ for *four* homogeneous magnitudes. Similarly, the results $A:C::B:C$ if and only if $A=B$ and $C:A::B:A$ if and only if $B=C$ are generalizations with the added constraint that A, B, C are homogeneous. The concepts of duplicate and triplicate ratio are **generalized** via the

[71] *Elements*, V, 8.
[72] *Elements*, V, 13.
[73] *Elements*, XII, 2.
[74] *Elements*, XII, 10.
[75] *Elements*, XII, 18.

simplification of the constraints on inputs. The operation "compounding ratios" is not adequately defined.

The propositions $A{:}B{::}C{:}D$ iff $AD=BC$ are **deleted** from pure ratio theory, and appear instead as applications to geometrical figures. The conceptual changes here are complex: the result is **specialized** via **complication** to the case where A, B, C, D are lines and AD, BC rectangles; at the same time, since the lines may be incommensurable or commensurable, the result is a **generalization** of the corresponding theorem for numbers. Similar remarks apply to the result $A{:}B{::}AC{:}BC$. Finally, new elements are **added**: the definition of "greater ratio", and the important techniques which involve this concept.

The generalized ratio concept applies to all homogeneous magnitudes, thus restoring the early assumptions of the arithmetical theory. The provision of an order relation and new techniques of approximation lead to new applications with the method of exhaustion.

The theory of proportion developed in Book V must be regarded as an almost complete success. Incommensurables as well as commensurables are accommodated by the definition. The standard results of the arithmetical proportion theory are also reformulated more generally, whether as theorems concerning ratios alone, or as applications of ratio theory to geometry. The theory of similar figures is re-established on a sound basis, and the exploitation of the order relation of ratios opens up a particularly fruitful line of research with the method of exhaustion. The difficulties and incompleteness which some commentators have attributed to the anthyphairetic theory do not occur in the Book V treatment. It is, moreover, a model of precise, abstract statement and proof—there are few unstated assumptions, and the treatment is quite general—again in contrast to what appears to have been the case in the older theories, where appeals to geometric construction were regularly made. The notion of compound ratio, it is true, is not given a satisfactory definition. However, this is a major fault only if one makes the assumption that the Greeks aimed at building a theory of real numbers, and thus required a serviceable definition of multiplication. We maintain that any such attribution is misguided.

Conclusion

We conclude by summarizing the changes we have identified in the Greek concepts of ratio and proportion that took place in reaction to the discovery of incommensurables, and by briefly discussing the nature of the explanation of these changes provided by Darden's strategies for anomaly resolution.

At the center of the older concepts of ratio and proportion was the assumption that whole numbers and their ratios could satisfactorily account for all the quantitative relationships encountered in science. Numbers were interpreted concretely, sometimes as the basic constituents of the universe; addition,

subtraction, multiplication, division, all had concrete interpretations. Around this notion of number the arithmetical concepts of ratio and proportion were constructed. With the discovery of incommensurable magnitudes, the basic assumption of this approach was abandoned. The concept of magnitude had to be generalized to account for quantitative relationships in geometry inexpressible in terms of whole numbers and their ratios. Interestingly, the notion of number did not follow this generalization of the concept of magnitude, but retained the restricted meaning of whole number. The loss of the concepts of ratio and proportion had far reaching consequences for Greek mathematics, and these appear to have been faced squarely. In the line of research which has come to be called geometrical algebra, the basic arithmetical operations were reinterpreted in geometrical terms, and a number of results which had previously been formulated with the help of ratio and proportion were restated without these troublesome concepts. The success of this venture was modest, and its costs relatively high: the loss of the power of concepts of ratio and proportion, and the strains involved in avoiding their use, must surely have been obvious.

Accordingly, we see evidence of two attempts to generalize the concepts of ratio and proportion to deal with all geometrical magnitudes, whether commensurable or not. The first of these relies on the adaptation of an arithmetical technique (anthyphairesis) to geometrical magnitudes. By grounding the concept of ratio in geometrical constructions, this approach deals directly with the problem of applicability: to have a ratio is no more than to be subject to a certain geometrical construction. By interpreting ratio geometrically, however, a number of complications enter in: the universal calculational power of the older arithmetical concepts is lost, subordinated to questions of geometrical possibility. General results (e.g. alternation of proportion) in the arithmetical theory are turned into a variety of special cases. The second generalization of ratio and proportion, that of Book V of the *Elements*, increasingly abstract and more technically flexible, avoids some of these problems while still meeting the challenge of the incommensurables. Magnitude is treated in great generality: only a small number of stated properties are required for the proofs of general results. The geometrical implementation of this concept, however, brings with it many of the difficulties which accompanied geometrical algebra and the anthyphairetic definition. Many of the results of the earlier arithmetical theory are lost, reappearing only as theorems concerning the application of proportion to geometrical figures. The operation "compounding" does not receive a satisfactory definition, and the completeness of the set of ratios is shown only for the case of line segments. Therefore, in spite of all its abstractness, one is still a long way from an independently defined, operational notion of continuous quantity like that found in later concepts of real numbers.

Our aim in this paper has been not only to describe the conceptual changes that took place in Greek concepts of ratio and proportion, but also to explain them. Our explanation showed how the various ways in which the Greeks attempted to deal with the problem of incommensurables constitute applications of the strategies for anomaly resolution identified by Lindley Darden. Although Darden makes no psychological claims about the strategies she identifies, we find it natural to construe them as mental mechanisms that thinkers use to solve conceptual problems. We hypothesize that the ancient Greek mathematicians, like the geneticists whose work Darden describes in detail, possess anomaly-driven mental procedures for altering concepts. This cognitive-science style of explanation is becoming increasingly established in the philosophy of science,[76] but is new to studies of conceptual change in mathematics.

With the important exception of dealing with equivalences (discussed in connection with tactic 5a concerning anthyphairesis), we found Darden's mechanisms for anomaly resolution to be adequate for generating the conceptual changes in Greek mathematics which followed the discovery of the incommensurables. This finding constitutes evidence that the development of mathematical knowledge involves some of the same mechanisms as the development of scientific knowledge. This is not to say that development is identical in both cases. Whereas anomalies in the sciences typically arise from experiment or observation, in the case we have studied the anomaly arose internally by the proof of the existence of incommensurables. Nevertheless, the strategies for dealing with anomalies seem to be fundamentally the same. Like science, mathematics requires strategies for conceptual change.

Acknowledgements—This research is supported by the Social Sciences and Humanities Research Council of Canada. We thank the anonymous referees for their helpful comments.

[76]See, for example, R. Giere, *Cognitive Models of Science*, Minnesota Studies in the Philosophy of Science, vol. 15 (Minneapolis: University of Minnesota Press, 1992); R. Giere, *Explaining Science: A Cognitive Approach* (Chicago: University of Chicago Press, 1988); P. Thagard, *Conceptual Revolutions* (Princeton: Princeton University Press, 1992); P. Thagard, *Computational Philosophy of Science* (Cambridge, MA: MIT Press/Bradford Books, 1988); P. Thagard, 'Mind, society, and the growth of knowledge', *Philosophy of Science* (in press).